

# Engineering students' conceptual understanding of the derivative in calculus

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**ABSTRACT:** The derivative is one of the important topics that engineering students encounter at university level. The research reported here, applied a theoretical framework combining the strengths of two major mathematics education theories in order to investigate the learning and teaching of the derivative. This article highlights some of the overall findings of this research with regard to students' understanding of the concept of derivative, and suggests applications of the framework to learning and teaching in undergraduate mathematics classrooms.

## INTRODUCTION

With regard to education in the field of engineering, many studies have pointed out that mathematical instruction for engineering students should not only include mathematical knowledge but also training in mathematical thinking [1][2]. Cardella used undergraduate and graduate engineering students, as well as engineers, as research targets for observation and interviews [1][2].

The results of that study showed that training in mathematical thinking is the most important task for university mathematics education and that it is of critical importance to develop mathematical thinking, as a type of mathematical literacy, of engineering students during their studies.

Not many studies have been conducted on how university education equips students with mathematical thinking and literacy. Using the concept of the derivative in calculus as an example, this study purposed to develop a calculus teaching module and assessment tool to cultivate mathematical thinking in freshmen engineering students. Two conceptual frameworks were used to form the theoretical basis of this study.

First, the Action-Process-Objects-Schema (APOS) theory was used to construct the development of the concept of derivatives. Second, representation theory provided conceptual tools to analyse the flexibility of using different representations and to understand the role played by the concept of derivatives. The APOS theory emphasises that conceptual formation works in stages. The construction of a complete concept or mental structure operates through four stages: action, process, objects and schema [3][4].

The concept of action is an individual and externally enabled conversion of concrete mathematical objects based on definite algorithms. Gradually, as individuals reflect more on their actions, they can internalise these actions into processes. At this point, the individual no longer needs to depend on a step-by-step operation but can use imagination.

When a mental process becomes a target for further manipulation and can be converted at will, this mental process is encapsulated and forms a detached context and a more formalised object. This object, then, becomes a target for further manipulation, and the process brings the individual onto a more abstract level. The entire process of transforming concrete objects to obtain more abstract new objects forms a schema. Through this cyclic process, a student's schema continually expands, and the manipulated objects become more and more abstract.

The knowledge acquired also grows increasingly formalised. After experiencing such a learning process, students will not have to build a new schema when processing similar problems: the original schema is sufficient to allow students to process another similar problem situation. In other words, the students will have generalised the schema.

In the mathematical learning process, representation is an essential tool used for expression of mathematical concepts, communication and thought. External representation refers to observable symbols, figures and tables, models, and images. Internal representation refers to the mental images constructed by a student. Students can use external representation to produce an internal representation of mathematical concepts. When the various changes in the internal representation of a mathematical concept and the functional relationships among these changes have been developed, the concept has been learned [5].

Therefore, a mathematical concept can be represented in multiple ways. Different forms of representation can be used to express or build the same concept, and each representation has advantages that make it superior to other representations. In discussing these advantages, Tall felt that graphics provide qualitative and comprehensive insight; quantities provide quantitative results and symbols provide a powerful capacity for manipulation [6]. However, according to Vinner, algebraic models are most commonly used in calculus to solve routine problems, as compared with graphical or visual models [7].

The APOS theory and representation theory allow researchers to examine the same phenomenon from two different but complementary viewpoints. In APOS theory, by using actions or processes of a representation to describe the theory, reflection on actions can produce meaningful viewpoints or properties, causing the actions to become internalised as processes. By integrating representation theory, researchers can clarify the role of these actions by emphasising the necessity of distinct viewpoints or properties.

The APOS theory can be used to describe the relationship between two objects in the same schema, or the relationships among objects, processes or actions with different representations. Table 1 shows the relationship between these two theoretical concepts, as well as how they were applied to the concept of derivatives investigated in this study.

Table 1: The framework of the derivative.

	Numerical representation	Symbolic representation	Graphical representation
Action	average rate of change	differentiation rule	slope of secant line
Process	linearisation	approximation	linearisation
Object	instantaneous rate of change	derivative of function	slope of tangent

Based on the different representations of functions and the concept of derivatives, this study distinguished three types of representations of derivative. The numerical representation of derivatives utilises the difference quotient and the rate of change. With regard to the numerical (table) expression of the function, differentiation can be viewed as a rate of change calculated to approach a certain value. The calculation is performed through the difference quotient of two close function values. The numerical representation of a derivative involves determination of an arbitrarily small difference quotient, and is known as a numerical derivative.

With regard to the graphical representation of the function, a derivative can be viewed as the calculation of the slope of the curve. Intuitively speaking, the slope of the curve is derived through amplification of a region of the curve, causing the region to be raised as a partial line or the slope of the tangent. This is known as a graphical derivative. With regard to the symbol representation of the function, derivatives are often expressed through algebraic rules. For example, the symbol representation of the function is  $y = x^3$ . The symbol of its derived function is:

$$\frac{dy}{dx} = 3x^2 \text{ or } f'(x) = 3x^2. \quad (1)$$

This is known as a symbol derivative. In this study, the use of the term derivative refers to any representation of a derivative. Use of the term symbol derivative indicates that the result (derived function) of symbol representation differentiation was used. If data are expressed through methods, such as the difference quotient and tables, numerical derivatives are those involved in calculation of the average rate of change or instantaneous rate of change. Graphical derivatives are those that relate to a secant line and the slope of a tangent.

## RESEARCH METHODOLOGY

The research subjects were 35 freshmen from a university engineering department. Due to space limitations, the test questions presented in this study consisted of only two differentiation problems, which are separately described below:

Problem 1:

Straight line L is the tangent of the graph of function  $y = f(x)$  at point (5,3), as shown in Figure 1, below. Please use the figure to a) calculate the value of  $f(5)$ ; b) calculate the value of  $f'(5)$ ; and c) estimate the value of  $f(5.1)$ .

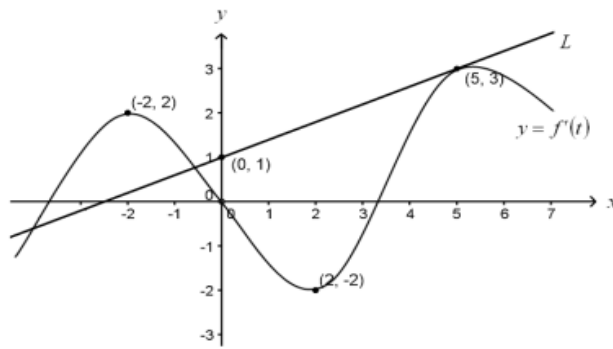


Figure 1: A straight line and a function.

Problem 2:

The figure below is the graph of the speed function  $y = f'(t)$  of Hsiao Ming's driving. Please answer the following questions:

1. Graph the accelerated function curve of Hsiao Ming's driving.
2. What is Hsiao Ming's instantaneous acceleration at  $t=4$ ?

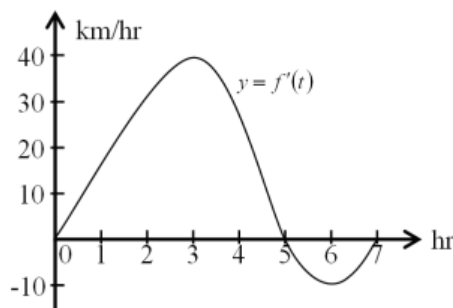


Figure 2: Graph of the speed function.

After the instruction on derivatives had been completed, a test was administered during class time. The test duration was 50 minutes. Students were required to show their problem-solving processes on paper and explain their reasoning. In the interview process, the problem was read out and students were shown their original test papers. After retracing their solution procedure, students were encouraged to describe their thinking and problem-solving processes. Several more specific questions were asked until a full picture of the student's mathematical thinking was gained. This process was repeated for each problem.

Using the classification methods of Vinner and Dreyfus, the students were classified according to their solution procedures used on the two test problems. The first review of the students' test papers confirmed the problem-solving classification of each student. In the second review of the students' test papers, the number of students in each category from the two groups was calculated. Additionally, to take a further step in understanding and interpreting the comprehension of the students with regard to the concept of derivatives, semi-structured interviews were conducted to clarify any ambiguities on the students' answers to the test questions.

## RESULTS

This section uses interview extracts to describe the comprehension of students with regard to the three representations of derivatives.

### Conversion of Numerical Derivatives into Graphical Derivatives and Graphical Representations of Functions

Question 1a tested students' ability to use the graphical representation of functions to determine the function value of  $f(x)$  at  $x = 5$ . The students' degree of comprehension was divided into three categories. Students in the first category were equipped with the concept of the process of a function and were able to successfully solve the problem. They linked the function with its graph and understood that the  $y$  coordinate of the point in the graph was the function value of the  $x$  coordinate.

The following interview extract shows that students in the second category attempted to find the algebraic expression of the function to input 5 to calculate  $f(5)$ . They used the coordinates of point  $(5, 3)$  and the information in the graph to

calculate the slope of the straight line and, then, calculated the linear equation. Lastly, they input 5 to obtain the correct answer of 3.

Researcher: *How did you figure out this problem?*

Jerry: *This point on the curve is also on the straight line.*

Researcher: *Hmm.*

Jerry: *We have (5,3) and (0,1), so the slope can be calculated.*

Researcher: *What do you want to do?*

Jerry: *I want to calculate the linear equation.*

Researcher: *Why?*

Jerry: *After calculating the linear equation, I can input 5 to solve for  $f(5)$ .*

Researcher: *Can you find the value of  $f(5)$  from the function graph?*

Jerry (thinks for approximately 12 seconds): *No, you have to calculate the linear equation.*

Perhaps it cannot be fully asserted that the conceptual formation of students in the second category stopped at the action level. However, it is obvious they were unable to use any type of process concept to solve this unique function graph problem. These students had to use an algebraic expression to input  $x = 5$ , showing they were unable to progress beyond the action concept of functions. A small number of students were classified into the third category. They assumed that the graphical equation was  $f(x) = ax^3 + bx^2 + cx + d$ , and then input the coordinates of the points on the graph to obtain the coefficient.

Question 1b tested the students' comprehension of the relationship between the derivative of a function at a certain point and the slope of the tangent at that point. In other words, the problem assesses whether students understand that the value of  $f'(a)$  is the slope of the tangent that passes point  $(a, f(a))$  on the function graph. Without the algebraic expression of the function, could students use only the graphical information to consider and calculate the derivative? The comprehension process of the students can be roughly divided into three categories. As with the above-described problem, the researcher found that the main difficulty hindering students from successful comprehension was the lack of a conceptual comprehension of functions at the process level.

Additionally, this study found that students might have been equipped with the process concept when thinking about specific functions (algebraic expressions and non-algebraic expressions) but often failed when it came to derivatives. Students in the first category had already developed comprehension of the relationship between derivatives and the slope of a tangent. Student Taylor, for example, could clearly explain the relationship between the two.

Researcher: *Explain how you handled this problem.*

Taylor: *L is the tangent, and  $f'(5)$  is the derivative of  $f(x)$  when  $x=5$ , as well as the slope of the tangent of the curve when  $x=5$ . The slope ... the slope has two points... (mumbles to self) ...2/5.*

Researcher: *How did you calculate it?*

Taylor: *The L tangent passes through points (5,3) and (0,1). I used a formula to calculate the slope of the straight line.*

Researcher: *Can you tell me why the derivative is the slope of the tangent?*

Taylor: *Because the tangent is the limit of the secant line and the slope of the tangent is the limit of the slope of the secant line; this is the definition of a derivative.*

Students in the second category had some understanding of the relationship between the derivative and the slope of the tangent; however, they were unable to clearly describe the equality between the two and had to use the tangent equation. In other words, these students intuitively felt they had to use an algebraic expression to calculate the derivative, showing that their understanding of the concept of derivatives remained at the action level.

Students in the third category were completely unable to clarify the relationships among the limit, the derivative and the slope of the tangent. Ben, for example, could determine from the function graph that the value of  $f(5)$  was 3 and knew that a derivative is a type of limit, but was unable to comprehend that a derivative is a special limit.

Students encountered difficulty in converting a numerical derivative into a graphical derivative when only the function graph was provided and not the algebraic expression. They were dependent on the algebraic expression to act as the medium.

### Converting a Numerical Derivative into the Graphical Representation of a Function

Through Question 1c, the researcher hoped to understand whether students were equipped with the concept of linearisation, and whether they could connect numerical derivatives and symbol derivatives. The problem-solving processes of the students can be divided into three categories. Students in the first category could use the value of  $f'(5)$  as the basis for linearisation and were equipped with the process concept of derivatives. The interview extract below uses Kevin as an example:

Researcher: *How did you solve this problem?*

Kevin: *The slope of the tangent at (5,3) is 2/5; in other words, when x increases by one unit, y increases by 2/5 of a unit.*

Researcher: *How do you know it is an increase rather than a decrease?*

Kevin: *2/5 is positive, so it is an increase.*

Researcher: *And then?*

Kevin: *Now the problem says that x increased by 0.2 units, so y increased by 0.04 units, and 3 plus 0.04 equals 3.04.*

Most of the students in the second category knew that the derivative is the slope of the tangent but estimated the value of  $f(5.1)$  based only on the function graph, thereby showing a lack of the process concept of derivatives.

Students in the third category used the right-hand limit for estimation. They thought that  $f(5.1)$  is the right-hand limit of  $f(x)$  when  $x = 5$ . Because the limit value is 3, the right-hand limit also equals 3. The following interview excerpt uses Amy as an example:

Researcher: *Can you explain how you calculated  $f(5.1)$ ?*

Amy: *I calculated that  $f(5)$  is 3, so  $f(5.1)$  is also 3.*

Researcher: *Why?*

Amy: *The limit value is 3. The values of the left and right-hand limits are equivalent.  $f(5.1)$  is the right-hand limit, so its value is also 3.*

Researcher: *Why is  $f(5.1)$  the right-hand limit?*

Amy: *The teacher explained it when teaching us about limits. When using a chart to determine limit values, 5.1 approaches 5 from the right trend of 5, so it is the right-hand limit.*

Students in the fourth category were similar to those in the second category for Question 1a: They inputted  $x$  as 5.1 into the equation to find  $f(5.1)$ .

#### Graphical Representation of Functions and Conversion of Symbol Derivatives into Graphical Derivatives

Question 2a required students to graph the acceleration function based on the graph of the speed function. The problem-solving processes of the students can be divided into four categories. Students in the first category could correctly solve the problem because they understood that the slope of the tangent is the acceleration and were able to graph the acceleration function based on the changes in slope of the tangent.

Students in the first category had internalised the concept that the graphical process determines the slope of the tangent process, and linked the concept to viewing the tangent as an object. These students drew the tangent on the graph of the speed function and could explain that the slope of the tangent demonstrated the results of differentiation; in other words, the derivative. They also knew that the derivative of the speed function was the acceleration function.

Students in the second category also drew the tangent on the graph of the speed function. The difference between these students and those of the first category was that they did not notice that the point of the horizontal axis was the point where the acceleration function curve passed the  $x$ -axis. Students in the second category only noticed the changes in the slope of the tangent and did not notice that the sign of the slope of the tangent changed from positive, to zero, to negative, and back to positive again.

Students in the third category noticed that the sign of the slope of the tangent changed from positive, to zero, to negative, and back to positive, but thought that each section of the slope of the tangent was constant. Students in the fourth category used the coordinates of the points on the curve to determine the graphical equation.

#### Conversion of Graphical Derivatives into Graphical Representation of Functions and Numerical Derivatives

Students encountered difficulties when calculating the instantaneous acceleration of the object at  $t = 4$ . The problem-solving processes of the students for Question 2b can be divided into four general categories. Students in the first category drew a tangent at  $t = 4$  and estimated the slope of the tangent according to the coordinates at the point of intersection between the tangent and the two axes. Andrew, the subject of the interview below, was classified into this category.

Researcher: *Why did you have to draw a tangent here?*

Andrew: *Because the question asks for the instantaneous acceleration of the object at  $t=4$ . Calculating the instantaneous acceleration requires calculation of the derivative, which requires calculation of the slope of the tangent.*

Researcher: *Okay, so how did you calculate the slope of the tangent?*

Andrew: *This tangent approximately passes these two points, (0,30) and (1.4,0). The calculated slope is then 21.4. So the instantaneous acceleration of the object at  $t=4$  is -21.4kph.*

Researcher: *Why is it negative?*

Andrew: *Because the speed is decreasing; it is deceleration.*

Students in this category had internalised the process of relating the slope of the tangent to the instantaneous rate of change, and connected the concept to viewing the tangent as an object. Students in the second category determined several points near point (4,30) and used the average acceleration to estimate the instantaneous acceleration. Sherry is an example of a student in this category.

Researcher: *Why did you select these points?*

Sherry: *I selected these points in order to calculate the average acceleration. The closer the points I select are to this (referring to point (4, 30)), the closer the average speed will be to the instantaneous acceleration.*

Researcher: *Okay. Did you think of any other methods?*

Sherry: *No.*

Students in this category had internalised the process of relating the average rate of change to the slope of the tangent but had not reached the level of materialisation. Students in the third category calculated the average speed from  $t = 0$  to  $t = 4$ , demonstrating that their conceptual formation had stopped at the action level. Students in the fourth category can be likened to the fourth category students from question 1a. They calculated the instantaneous acceleration according to the differential equation derived from the previous question.

## CONCLUSIONS

This study used teaching experiments to investigate the conceptual understanding of students after they had been taught about the concept of derivatives. The researcher found that students could learn about the concept of derivatives through development of different representations. Due to the use of these representations, the comprehension of 81% of students in the representation group with regard to the concept of derivatives closely approached the object level.

The teaching module began with the average rate of change in a quantified discrete context and, then, commenced to the conversion of the representation using graphical derivatives and lastly numerical derivatives. Research results showed that this type of module is appropriate for developing comprehension of the relationship between functions and derivatives. The results of this study support the position of Krutetskii [8]. When students favour one type of mathematical thinking mode or are dominated by one type of representation, this limits their mathematical thinking and leads to different learning difficulties.

With regard to methods involving concrete and dynamic imagery, Presmeg indicated that concrete imagery is a source of learning difficulty for students [9]. However, on a certain level, dynamic imagery is effective in depicting changes in the slope of a tangent and conversion graphs. Through introduction to various types of representation, students could integrate various modes of thought to deepen their comprehension of the concept of derivatives and link this comprehension to their individual representation and imagery, thereby forming conventions [10]. The results of this study can both confirm and challenge the modes of thinking preferred by students and enrich calculus instruction.

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